Reggeized Double-Peripheral-Model Analysis of Three-Body Final-State Processes: Application to $pp \rightarrow p\pi^-\Delta^{++}$ at 28.5 GeV/c and other Reactions*

Edmond L. Berger†

Lawrence Radiation Laboratory,‡ University of California, Berkeley, California 94720 and Physics Department, Brookhaven National Laboratory, Upton, New York 11973

(Received 31 October 1968)

Use of a double-Regge-pole-exchange model to describe peripheral three-body final-state processes, especially at near-threshold values of the invariant mass of a pair of final-state particles, is described and discussed. The fact is stressed that the model provides an understanding of the entire reaction, yielding distributions in all kinematical variables. As an example, results are presented from a detailed comparison of the predicted model distributions with data from $pp \rightarrow p\pi^-\Delta^{++}$ at 28.5 GeV/c. One simple diagram involving only Pomeranchon and pion exchanges is employed. Good agreement is obtained with the experimental distributions in invariant masses, momentum transfers, and various angles; in particular, the enhancement near 1460 MeV in the $(\pi^-\Delta^{++})$ mass spectrum is well fitted. Recent applications of the model to the A_1 region in $\pi N \rightarrow \pi\rho N$ are reviewed. Also explored is the potential for extracting information on the pion trajectory from three-particle final-state reaction data.

I. INTRODUCTION

EXTENSION of the Regge-pole-exchange model from the quasi-two-body domain to peripheral inelastic processes involving several particles in the final state has been studied by several groups in recent years,[1–4] and phenomenological fits to processes with three particles in the final state have been achieved.[2,4–9] Some of these fits have been carried out within the strict limits of applicability of the Regge-pole model, namely, using only events in that region of the Dalitz plot where the final-state two-particle invariant masses are relatively large. Whereas this limitation can evidently be justified by invoking the origins of Regge theory, it has the unavoidable drawback, at presently accessible energies and current bubble-chamber experiment statistics levels, of leaving one with very few events to study.

On the other hand, recent work with finite-energy sum rules[10] in two-body scattering demonstrates that the parametrization appropriate to high-energy scattering does provide an adequate semilocal-average description of low-energy phenomena. Thus one is motivated to invoke the same type of result in the multiparticle domain and to suggest that a double-Regge-pole-exchange model may also, in restricted circumstances, be analytically continued below its designed region of evident validity and provide useful results when one or both two-particle invariant masses become small.[11]

Some results of such an extension have already been described by the author and successful fits to data from several reactions have been published.[5–7] In this paper additional characteristic features of the double-Regge approach are noted and compared with results from non-Reggeized Deck-type models.[12] An important aspect of the double-Regge approach which deserves emphasis is that it provides an understanding of the entire three-body final-state reaction; it yields distributions in all relevant kinematical variables, not merely fits to the Dalitz plot or to one Chew-Low plot.

In Sec. II, the double-Regge-pole exchange hypothesis for general three-particle final-state reactions in the form given by Bali, Chew, and Pignotti[1] is reviewed and, in Sec. III, the rationale for extending its application to near-threshold values in one of the final-state two-particle invariant masses is made explicit. These sections are intended to be fairly general and should provide a basis for use of the model in the analysis of an arbitrary three-particle reaction.

A particular example of the use of a double-Regge model at high energy is focused upon in Sec. IV. The reaction $pp \rightarrow p\pi^-\Delta^{++}$ at 28.5 GeV/c is discussed in some detail and distributions calculated from the double-Regge model are compared with the several

[1] N. F. Bali, G. F. Chew, and A. Pignotti, Phys. Rev. Letters 19, 614 (1967); Phys. Rev. 163, 1572 (1967). An extensive list of references to earlier work may be found in G. F. Chew and A. Pignotti, Phys. Rev. 176, 2112 (1968).
[2] H. M. Chan, K. Kajantie, and G. Ranft, Nuovo Cimento 49, 157 (1967); H. M. Chan et al., ibid. 51, 696 (1967). For a partial review of the double-Regge-pole approach, including references to phenomenological fits, see articles by Chan Hong-Mo and S. Ratti in Topical Conference on High-Energy Collisions of Hadrons (CERN, Geneva, 1968).
[3] F. Zachariasen and G. Zweig, Phys. Rev. 160, 1322 (1967); 160, 1326 (1967).
[4] R. G. Roberts and G. M. Fraser, Phys. Rev. 159, 1297 (1967).
[5] E. L. Berger, Phys. Rev. 166, 1525 (1968).
[6] E. L. Berger et al., Phys. Rev. Letters 20, 964 (1968).
[7] E. L. Berger, Phys. Rev. Letters 21, 701 (1968).
[8] A. M. Cnops et al., Phys. Rev. Letters 21, 1609 (1968); Ian O. Skillicorn (private communication).
[9] M. L. Ioffredo et al., Phys. Rev. Letters 21, 1212 (1968).
[10] R. Dolen, D. Horn, and C. Schmid, Phys. Rev. 166, 1768 (1968); K. Igi and S. Matsuda, Phys. Rev. Letters 18, 625 (1967).

[11] G. F. Chew and A. Pignotti, Phys. Rev. Letters 20, 1078 (1968); G. F. Chew, Comments Nucl. Particle Phys. 2, 74 (1968).
[12] L. Stodolsky, Phys. Rev. Letters 18, 973 (1967); M. Ross and Y. Yam, ibid. 19, 546 (1967); U. Maor, Ann. Phys. (N. Y.) 41, 456 (1967); L. Resnick, Phys. Rev. 150, 1292 (1966). References to previous work may be found in these papers.

experimental distributions.[13] A choice is made of an appropriate unique double-exchange diagram on the basis of certain kinematic selections and then the matrix element is parametrized in terms of two constrained parameters: a scale constant $s_{0\pi}$ and the slope of the pion trajectory, assumed linear. By adjusting the two parameters, a good fit to the data is obtained; the pion-trajectory slope determined in this fashion is $\alpha_{\pi}'=1.2$ GeV^{-2} and the scale constant is $s_{0\pi}=0.7$ GeV^{2}, both quite acceptable values and in agreement with the parameters determined in an earlier modified double-Regge-model fit to data on the same reaction at 6.6 GeV/c.[6]

The invariant mass allowed to approach threshold in this calculation is that of the $(\pi^-\Delta^{++})$ system. What emerges is an acceptable fit to the near-threshold 1460-MeV enhancement in this mass spectrum without the necessity of invoking a resonance interpretation of the phenomenon.

In Secs. IV F and V, the more general features of the double-Regge-model distributions and particular phenomenological consequences of the model are emphasized and contrasted with results of the non-Reggeized Deck-type models.

Section VI is devoted to a discussion and interpretation of the results. Included are references to recent work in which the model was applied to the processes $\pi^-p \to \pi^-\rho^0 p$ and coherent $\pi^+d \to \pi^+\rho^0 d$, and agreement achieved with the experimentally observed $(\pi\rho)$ mass enhancement in the A_1 region.[8,9]

For all reactions studied, a linear pion trajectory with an average slope of approximately 1.0 (GeV/c)$^{-2}$ seems to yield best agreement with the data; the sense in which the pion-trajectory slope is determined from this analysis is discussed in Secs. IV B and IV F.

II. DOUBLE-REGGE MODEL

In this section the hypothesis of the double-Regge-pole-exchange model[1-4] is briefly reviewed for the general-mass, three-body final-state process:

$$m_1+m_2 \to \mu_1+\mu_2+\mu. \qquad (1)$$

Commonly, the assumption of the Regge approach is that the model is strictly applicable only in that central region of the Dalitz plot where the invariant mass of each final-state pair of particles is "large." Whether one uses the Toller-variable route[1] or a procedure involving a double Sommerfeld-Watson transformation, this restriction obtains directly because only when the invariant mass is large does the asymptotic expansion
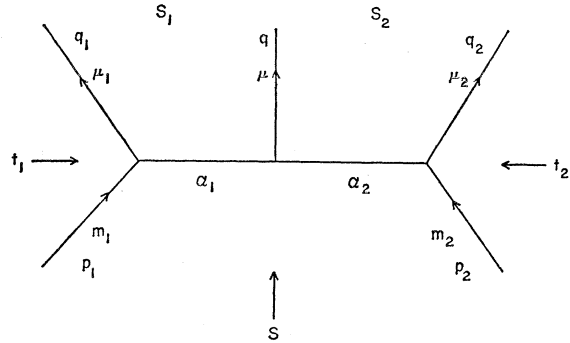

FIG. 1. General double-Regge-pole-exchange diagram for the process $m_1+m_2 \to \mu_1+\mu+\mu_2$. The p_i and q_i are four-momenta and the m_i and μ_i are masses. $s_1=(q_1+q)^2$, $s_2=(q_2+q)^2$, $t_2=(q_2-p_2)^2$, $t_1=(q_1-p_1)^2$, $s=(p_1+p_2)^2$. The α_j denote Regge trajectories.

yielding a $(s_i)^{\alpha_i}$-type dependence on the subenergy variable s_i result naturally. The hypothesis asserts that reaction (1) is dominated by a *sum* of diagrams of the form given in Fig. 1.

Generally stated, in the sum there is a diagram for each possible ordering at the three vertices of the final set of particles relative to the initial pair and, given an ordering, for each unique pair of trajectories whose quantum numbers allow coupling to the external particles. Although this statement means that the analysis for a given process will require many diagrams, limiting consideration to a restricted kinematic region reduces the number of important diagrams. This procedure will be discussed below, after the general situation is pursued here.

For the sake of clarity, in this section all the external particles in the diagram of Fig. 1 will be considered spinless; a helicity-amplitude approach for the general-spin case may be found in Ref. 4. For the specific diagram given in Fig. 1, with the mass and trajectory labeling indicated, the invariant amplitude is[1,2,14,15]

$$\mathfrak{M}(s,s_1,s_2,t_1,t_2) \approx F_1(t_1)[(s_1\cdots)/s_{10}]^{\alpha_1}F_3(t_1,t_2,\omega)$$
$$\times[(s_2\cdots)/s_{20}]^{\alpha_2}F_2(t_2). \qquad (2)$$

In this expression, each of the functions $F_i(t_i)$ contain as factors (a) the propagator function for trajectory $\alpha_i(t_i)$, (b) the signature factor for trajectory α_i, (c) necessary kinematic factors associated with the coupling of m_i and μ_i to trajectory α_i, and (d) the reduced-residue function associated with the (m_i,μ_i,α_i) coupling. Insofar as they can be determined from Regge fits to quasi-two-body processes, these four factors are, in principle, all known quantities.

[13] The data were derived from a Brookhaven National Laboratory Bubble Chamber Group study of $\bar{p}p$ interactions as reported in P. L. Connolly *et al.*, Brookhaven National Laboratory Report No. 11980 (unpublished); W. E. Ellis *et al.*, Brookhaven National Laboratory Report No. 12673 (unpublished), submitted to the Fourteenth International Conference on High Energy Physics, Vienna, 1968; W. E. Ellis *et al.*, Phys. Rev. Letters **21**, 697 (1968).

[14] The metric use here is specified by $q \cdot q = q_0{}^2 - \mathbf{q} \cdot \mathbf{q} = m^2$, where q is a four-vector.

[15] At fixed s, the invariant amplitude for reaction (1) is a function of four independent variables which are taken here to be the set of Lorentz-invariant quantities defined in Fig. 1: two subenergy variables s_1 and s_2 are two momentum-transfer variables t_1 and t_2. The notation $M(\mu_1\mu_2)$ denotes the invariant mass of the pair of particles μ_1 and μ_2.

In Eq. (2), the factors $[(s_i \cdots)/s_{i0}]^{\alpha_i}$ provide the characteristic Regge form of the amplitude; the s_{i0} are scale constants and, judging from two-body fits, $s_{i0} \approx 1$ GeV^2. The quantity $(s_i \cdots)$ is the numerator of the $\cosh \xi_i$ variable of Bali, Chew, and Pignotti,[1] the denominator of which is absorbed into $F(t_i)$ and $F(t_1, t_2, \omega)$ as a kinematic singularity. Explicitly, for the general-mass configuration of Fig. 1,

$$(s_1 \cdots) = s_1 - t_2 - m_1{}^2 + \tfrac{1}{2} t_1{}^{-1}$$
$$\times (\mu_1{}^2 - m_1{}^2 - t_1)(\mu^2 - t_1 - t_2). \quad (3)$$

An analogous expression for $(s_2 \cdots)$ is obtained by interchange of the subscripts 1 and 2 in Eq. (3). The remaining factor in Eq. (2), the function $F(t_1, t_2, \omega)$, describes the coupling of the two Reggeons α_1 and α_2 at the central vertex to the emitted particle of mass μ. The variable ω, a natural rotation angle in the Toller-analysis approach may be defined as[1,2]

$$\cos\omega = (\mathbf{p}_1 \times \mathbf{q}_1) \cdot (\mathbf{p}_2 \times \mathbf{q}_2)/|\mathbf{p}_1 \times \mathbf{q}_1||\mathbf{p}_2 \times \mathbf{q}_2|, \quad (4)$$

where the three-vectors $\mathbf{q}_i$ and $\mathbf{p}_i$ are evaluated in the Lorentz frame in which $\mathbf{q} = 0$, i.e., the rest frame of the particle emerging from the central vertex. As this equation indicates, ω is the angle between the normals to the $(\mathbf{p}_1, \mathbf{q}_1)$ and $(\mathbf{p}_2, \mathbf{q}_2)$ production planes, as viewed from the frame of reference in which $\mathbf{q} = 0$.

The reaction amplitude $\mathfrak{M}$ on the left-hand side of Eq. (2) has been expressed as a function of the complete set of independent variables (s, s_1, s_2, t_1, t_2). From the purely kinematic point of view, ω is not a variable independent of that set; as pointed out most explicitly by Bali, Chew, and Pignotti,[1] ω is complementary to the total energy variable s. Nevertheless, the range of ω, at fixed s, is not limited: It extends from 0 to π. Moreover, it is meaningful, even at fixed s, to express the right-hand side of Eq. (2) in terms of ω because the two-Reggeon–one-particle vertex function may, in general, have a dynamical dependence on that variable. To be sure, in order to perform an explicit calculation at fixed s, the asserted dynamical dependence on ω must be transformed into a dependence on the chosen set of independent variables. By the same token, because the kinematic relationship between ω and any one of the set (s, s_1, s_2, t_1, t_2) involves all members of the set, there is no reason to expect the distribution in the variable ω to be isotropic, even if the function $F(t_1, t_2, \omega)$ should be entirely free of explicit ω dependence.

The general multi-Regge analysis does not presently specify the dependence of this central-vertex function on ω and on the t_i; barring detailed model calculations, this dependence must be sought out phenomenologically. This situation is similar to that in the two-body Regge approach in which one determines the t dependence of the reduced-residue functions phenomenologically.

For a specific, peripheral three-particle final-state reaction, given the set of relevant doubly peripheral diagrams [and their amplitudes in the form of Eq. (2)],

one presumably has in the double-Regge model a complete description of the physical process. Specifically, one should be able to produce an adequate fit for the distributions in *all* possible momentum-transfer, final-state two-particle invariant-mass and angular variables. It should be noted that this approach is considerably more ambitious than, for example, the Deck-type[12] models which have for the most part been limited to a description of the Chew-Low plot in one momentum-transfer and one invariant-mass combination for a given reaction. The process $pp \to p\pi^- \Delta^{++}$ is examined later in this paper as an example of the complete fit stressed above.

The remarks of the previous paragraph are subject, of course, to the qualification that the Regge model is usually considered applicable only when the various two-particle final-state invariant-mass values are large. Section III is devoted to a discussion of the reasoning behind extending the Regge model to cover the entire Dalitz plot.

III. EXTENSION TO SMALL SUBENERGIES

Motivation to apply the double-Regge-pole-exchange model below its region of orthodox validity stems from various sources. Empirically, it is interesting to contemplate for a typical process the fraction of data remaining after imposition of the orthodox restriction that all final-state two-particle invariant masses be "large." Consider, for example, the reaction $pp \to pn\pi^+$ at incident proton lab momentum 28.5 GeV/c.[13] Barely 4% of the events are left after elimination of those for which either the mass $M(n\pi^+) < 2.0$ GeV or $M(p\pi^+) < 2.0$ GeV. At current high-energy bubble-chamber experiment statistics levels, therefore, fewer than 50 events would be available for Regge-model analysis.[16] Whereas consistency of the model with the data of this limited sample is evidently essential, it is also valuable to attempt to broaden the scope of inquiry.

An analytical understanding of the fact that approximately 96% of the data from $pp \to pn\pi^+$ at 28.5 GeV/c is concentrated in that segment of the Dalitz plot where either $M(n\pi^+) < 2.0$ GeV or $M(p\pi^+) < 2.0$ GeV is easily given in terms of a doubly peripheral picture. This issue has been explored quantitatively by various researchers, often in the context of the Deck effect.[12] An explicit statement in terms of Fig. 1 is the following: Although, on purely phase-space grounds, each s_i may range from $(\mu + \mu_i)^2$ to $(s^{1/2} - \mu_j)^2$, $j \neq i$, the graph in Fig. 1 implies that both t_i are heavily weighted towards their maximum (kinematic) limiting values (e.g., t_i weighted near 0 if $\mu_i > m_i$) which, in turn, substantially kinematically distorts the phase-space spectrum in the s_i. Indeed, the restriction of either t_i to values near its absolute kinematic limit has the effect of enhancing small values of both s_i. Consequently, there is an in-

---

[16] About 500 events appear necessary for an adequate study of three-body final-state production models at fixed s.

compatibility between the assertion of double periph-
erality and the orthodox requirement of a multi-
Regge model which would have the masses of all pairs
of final-state particles large.

Relaxation of the restriction to large s_i finds support
in the many recent successful applications of finite-
energy sum rules, in two-body reactions, in which the
low-energy direct-channel resonance approximation
has provided a good description of various cross-channel
Regge trajectories.[10] The import of this development
for three-particle reactions has been emphasized by
Chew and Pignotti.[11] What emerges is the suggestion
that in using the double-Regge model one should expect
to achieve a reasonably good semilocal-average descrip-
tion of the data over the complete spectrum of s_i
values. Certainly, sharp resonancelike detail cannot
result, but gross features of the s_i and other distribu-
tions should be well reproduced. Particularly interesting,
therefore, from the point of view of phenomenological
application of the model, are investigations of those
particular three-body reactions which either display
essentially no resonant effects in any subenergy[17]
variable, or display fairly structureless, broad enhance-
ments at low invariant-mass values for one pair of
final-state particles. Several examples of this latter
type will be discussed later in this paper.

Finally, the multi-Regge approach to several-
particle production[1] would be facilitated by a convincing
demonstration in the three-particle arena that the use
of a double-Regge approach at small subenergies is in
acceptable agreement with experiment, especially in
the sense of generating a suitable average over the
resonance region. The demonstration would help to
justify a multi-Regge description of multiple production
which ignores resonances and concentrates rather on
computing with diagrams that have only stable par-
ticles in the final state.

In order to restrict the number of diagrams treated,
the approach taken here was to relax the orthodox
Regge-theory limitation in the case of one subenergy
variable only. An example of a calculation of this type
is given in Sec. IV.

IV. APPLICATION TO $pp \rightarrow p\pi^- \Delta^{++}$

A double-Regge-model analysis for the reaction
$pp \rightarrow p\pi^- \Delta^{++}$ is described in this section in a manner
general enough to allow similar application, for example,
to $pp \rightarrow \pi pp$, $\pi p \rightarrow \pi\pi p$, $\pi p \rightarrow \pi\rho p$, $Kp \rightarrow \pi K\Delta$,
$Kp \rightarrow \pi K^* p$, and analogous coherent processes. Pre-
dictions of the Regge model are compared here with
experimental distributions obtained by a Brookhaven
group studying proton-proton interactions at 28.5
GeV/c.[13]

---

[17] An example of this type has been studied by G. Alexander
et al., Phys. Rev. 177, 2092 (1969).

A. Choice of Diagrams

The essential simplifying conclusion reached in this
section is that by limiting the study to events for which
$M(p\pi^-) > 2.0$ GeV, one may adequately represent the
data with a unique double-exchange diagram, given
in Fig. 3(a), employing Pomeranchon and pion ex-
changes only. Other possible diagrams are judged to
provide contributions of secondary importance.

Since the incident particles in $pp \rightarrow p\pi^- \Delta^{++}$ are
identical, there are a priori only three generic types of
double-Regge-pole-model diagrams which can be written
for this process, differing according to which final-state
particle is coupled at the central two-Reggeon–one-
particle vertex. The quantum number structure of two
of these types, with either π or Δ emerging from the
central vertex, is such that a Pomeranchuk trajectory
(P) can be accommodated as one of the pair of ex-
changed Reggeons. The third type of diagram, with a
proton emitted at the middle vertex, cannot admit P
exchange.

Separate consideration of diagrams with and without
P exchange is justified by various arguments. One line
of argument is based on the fact that for a diagram
containing P exchange the total cross section σ is
approximately s-independent, whereas σ falls with s
if P exchange is not present. Another line of reasoning
relates to the fashions in which various exchanges
populate the allowed ranges of the various subenergies.
Because of the characteristic $(s_i/s_0)^{\alpha_i}$ subenergy de-
pendence in the amplitude, the contribution of the
Pomeranchuk trajectory ($\alpha_P \approx 1$) will dominate when
the associated subenergy is large. In fact, the P will
most effectively overcome the preference, discussed
earlier, of double-peripheral diagrams for small sub-
energies. On the other hand, the lower lying a given
trajectory is, the closer to threshold in the associated
subenergy will its contribution be felt.

In regard to the specific process under study here, the
restriction to events with $M(p\pi^-) > 2.0$ GeV should be
sufficient to justify the disregard in first approximation
of all diagrams other than those containing P exchange.
The two remaining diagrams of interest are drawn in
Fig. 2, where the meson M and baryon B exchanges must
now be specified.

Quantum number requisites at the central vertex of
Fig. 2(a) demand that M have G parity (-1) and isospin
1; the obvious candidates are π and A_1. The standard
nearest-singularity argument suggests that the π con-
tribution is dominant over that of A_1 at small values of
the momentum transfer to the Δ^{++}. In addition, a
comparison of the on-mass-shell πp elastic and $\pi p \rightarrow Ap$
cross-sections shows that the strength of the $\pi\pi P$ coupl-
ing is a factor of 10 greater than the πAP coupling; be-
cause such coupling constants enter at the middle vertex
of the two diagrams, π exchange would appear to be
substantially favored. For Fig. 2(b), the argument is

similar, with the conclusion being that the baryon B is the Δ^{++} itself.

The diagrams of Fig. 2 contribute to overlapping regions of phase space, and a meaningful separation of their contributions requires limitations on t_Δ and t_π. Kinematically, although the maximum value of t_π is given by $(m_p - m_\pi)^2 \approx +0.6$ GeV^2, a discussion based on the distance from the baryon pole can be used to argue that the magnitude of the contribution of Fig. 2(b) should be suppressed in relation to that of Fig. 2(a); interference questions are more delicate. The easily derived relationship,

$$t_\pi + t_\Delta = t_p - s_{\pi\Delta} + m_p^2 + m_\Delta^2 + m_\pi^2, \qquad (5)$$

indicates, however, that interference is important only in regions of phase space distant from both poles, for example, where $t_\Delta < -0.5$ GeV^2.

Finally, to facilitate comparison with the limited quantity of data, only the pion–Pomeranchon diagram given in Fig. 3(a) was retained in the detailed computations. Because of the approximations discussed thus far, certain qualifications as to the expected results are evident. One should not expect to compute with this one diagram the entire observed cross section. Moreover, in adjusting free parameters associated with Fig. 3(a) in an attempt to fit data, one will perforce be generating some type of average over the background associated with neglected diagrams.

B. Matrix Element

The invariant amplitude for the diagram in Fig. 3(a) will be written as a function of the set of five Lorentz-
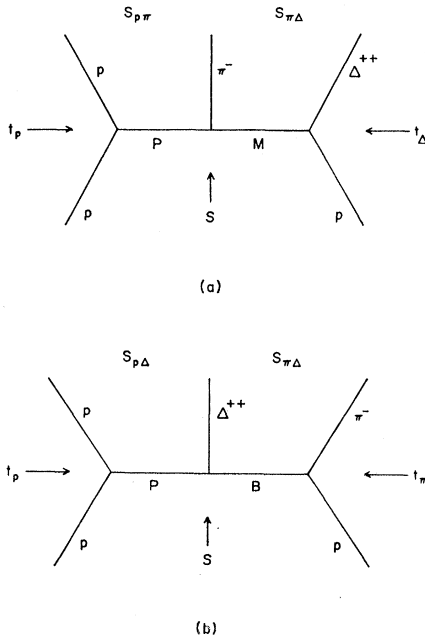

(a)

(b)

FIG. 2. Diagrams containing Pomeranchuk-trajectory (P) exchange which contribute to $pp \to p\pi^-\Delta^{++}$; M denotes a meson-type and B a baryon-type trajectory.
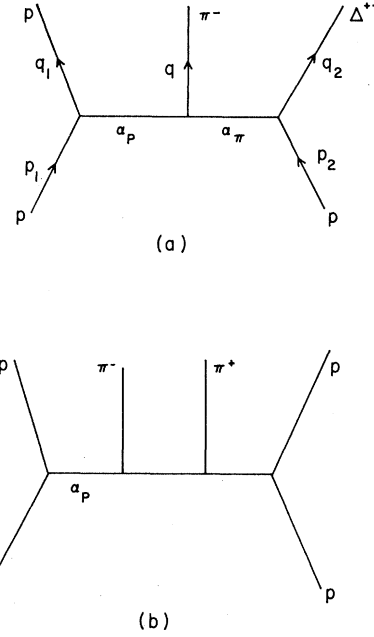

(a)

(b)

FIG. 3. (a) Dominant double-Regge-pole-exchange diagram for the reaction $pp \to p\pi\Delta^{++}$; α_P denotes the Pomeranchuk and α_π the pion trajectory. The p_i and q_i are four-momenta. $s_{p\pi} = (q_1 + q)^2$, $s_{\pi\Delta} = (q_2 + q)^2$, $t_\Delta = (q_2 - p_2)^2$, $t_p = (q_1 - p_1)^2$, $s = (p_1 + p_2)^2$. (b) Triple-exchange four-particle final-state diagram representing background contribution.

invariant quantities defined in terms of the four-momenta p_i and q_i as $s = (p_1 + p_2)^2$, $t_\Delta = (q_2 - p_2)^2$, $t_p = (q_1 - q_1)^2$, $s_{\pi\Delta} = (q + q_2)^2$, and $s_{p\pi} = (q + q_1)^2$. The differential cross section for the process is

$$d\sigma = (2\pi)^{-5}(4F_I)^{-1}(\sum |\mathfrak{M}|^2)d\varphi_3, \qquad (6)$$

in which F_I equals the proton mass times the incident proton lab momentum, and $d\varphi_3$ is the differential element of phase space.

The complexities of spin analysis were set aside by the adoption of the double-Regge-pole hypothesis for the absolute square of the invariant amplitude $\mathfrak{M}$ summed over final spins and averaged over initial spins.[1,2] This effective neglect of spin dependence has the practical consequence of reducing the number of parameters involved in describing the momentum-transfer (t_p and t_Δ) structure of the amplitude. In the paragraphs below, a description is given of the fashion in which this spin-averaged momentum-transfer dependence was specified. The explicit dependence of the amplitude on the subenergy variables (and thus on the trajectories) is not affected by the averaging, however, since each helicity amplitude[4] contains a common factor $(s_{\pi p} \cdots)^{\alpha_P}(s_{\pi\Delta} \cdots)^{\alpha_\pi}$ which can be extracted to provide the over-all dependence on the subenergies.

The discussion of the parametrization of $\sum |\mathfrak{M}|^2$ is based upon the functions F_i of Eq. (2). Because the model yields no prescription for the ω dependence of $F_3(t_\Delta, t_p, \omega)$, such explicit variation was initially assumed

absent here. This assumption seems justified by the resulting fits, especially by the fit to the distribution in itself, shown in Fig. 6(b).

In the attempt to reduce the number of free parameters, information obtainable from quasi-two-body reaction studies was incorporated into the specification of the t_Δ and t_p dependences of $(\sum|\mathfrak{M}|^2)$. Because fits[18] to two-body data conclude that the Pomeranchuk trajectory has little slope, if any, the P was chosen here to be a fixed singularity, $\alpha_P = 1.0$. Thus there is no P propagator to be discussed. For definiteness, the scale constant s_{0P} was set equal to 1.0 GeV^2, and then the remaining t_p dependence of the amplitude [in the product $F_1(t_p)F_3(t_p,t_\Delta)$ of Eq. (2)] was parametrized as an exponential function determined from high-energy πp elastic scattering data, as follows: In the limit $t_\Delta \rightarrow m_\pi^2$, t_p is simply the invariant-momentum-transfer variable for πp elastic scattering and the function $F_3(t_p,t_\Delta)$ becomes an external residue; thus, in that limit, the product $F_1(t_p)F_3(t_p,t_\Delta)$ is proportional to the t dependence of the amplitude for the elastic πp process. Under the assumption (1) that for small t_Δ, the t_p dependence of the product $F_1(t_p)F_3(t_p,t_\Delta)$ differs little from the on-shell case, and (2) that the Pomeranchuk trajectory is responsible for the πp elastic diffraction peak,[19] it is appropriate to set $|F_1(t_p)F_3(t_p,t_\Delta)|^2 \approx F_3'(t_\Delta)e^{8t_p}$, where $F_3'(t_\Delta)$ is an unknown slowly varying function.

The discussion of the previous paragraphs may be summarized in the following expression for the amplitude[20] corresponding to the diagram of Fig. 3(a):

$$\sum|\mathfrak{M}|^2 = N_0 G_\pi(t_\Delta)[(s_{\pi\Delta}\cdots)/s_{0\pi}]^{2\alpha_\pi}(s_{\pi p}\cdots)^2 e^{8t_p}. \quad (7)$$

Details of the π coupling to $p\Delta$ as well as the π propagator and the residual middle vertex t_Δ dependence, previously denoted by $F_3'(t_\Delta)$, are incorporated into $G_\pi(t_\Delta)$, and N_0 is the over-all normalization constant. Here

$$(s_{\pi\Delta}\cdots) = s_{\pi\Delta} - t_p - m_p^2 + \tfrac{1}{2}t_\Delta^{-1}(m_\Delta^2 - m_p^2 - t_\Delta)$$
$$\times (m_\pi^2 - t_p - t_\Delta), \quad (8)$$

$$(s_{\pi p}\cdots) = s_{\pi p} - t_\Delta - m_p^2 - \tfrac{1}{2}(m_\pi^2 - t_p - t_\Delta). \quad (9)$$

The characteristics of the pion trajectory are, at present, an issue of intense investigation in two-particle scattering and, as such, it is hardly possible to present a well-established expression for $G_\pi(t_\Delta)$. In this study the most natural assumptions consistent with Regge-theory phenomenology were made. A linear pion trajec-

tory was adopted,

$$\alpha_\pi = (t_\Delta - m_\pi^2)\alpha_\pi', \quad (10)$$

with the constant slope α_π' left as an adjustable parameter; $G_\pi(t_\Delta)$ was assumed to be given entirely by the product of the Reggeized propagator and signature factor with no other vertex structure:

$$G_\pi(t_\Delta) = (\pi\alpha_\pi')^2/2(1 - \cos\pi\alpha_\pi). \quad (11)$$

It will be noted that for $t_\Delta \rightarrow m_\pi^2$, $G_\pi(t_\Delta)$ approaches the elementary OPE (one-pion-exchange) expression: $G_\pi(t_\Delta) \rightarrow (t_\Delta - m_\pi^2)^{-2}$.

The expression for $G_\pi(t_\Delta)$ in Eq. (11) is not appropriate for large values of $|t_\Delta|$ [i.e., $|t_\Delta| > 1.0$ (GeV/c)^2] because, as it stands, it develops poles in the physical region at $\alpha_\pi = -2$, -4, etc. These unphysical poles could be eliminated by restoring to the right-hand side of Eq. (11) the factor $[\Gamma(1+\alpha)]^{-2}$, sometimes used in Regge-pole phenomenological fits.[21] However, because pion exchange is dominant only for small t_Δ, a cutoff in the integration over the t_Δ variable (and in the data) is appropriate here for physical reasons, and the difficulty is thus avoided.

Two adjustable parameters which may be varied to achieve a fit to the shapes of the several experimental distributions are embodied in Eqs. (7) and (10). They are *both associated with the pion-exchange* aspect of the diagram and are the slope α_π' of the pion-trajectory function and the scale constant $s_{0\pi}$.

In one-Regge-pole-exchange fits to quasi-two-body reaction data over a range of s values, it is possible to isolate the trajectory function $\alpha(t)$ from the remaining t dependence of the amplitude by fitting the differential cross section, at fixed t for various s, to an expression of the form

$$d\sigma/dt = f(t)(s/s_0)^{2\alpha(t)-2}. \quad (12a)$$

Technically, a similar program could be undertaken in this three-body reaction to separate the α_π' and $s_{0\pi}$ dependences of the amplitude; its results are inconclusive because of the limited quantity of data, however. The procedure requires performing the integrations over the $(s_{\pi p}\cdots)$ and t_p variables in Eqs. (6) and (7) to obtain a doubly differential cross section in the $s_{\pi\Delta}$ and t_Δ variables,

$$d^2\sigma/dt_\Delta d(s_{\pi\Delta}\cdots) = G_\pi(t_\Delta)[(s_{\pi\Delta}\cdots)/s_{0\pi}]^{2\alpha_\pi}$$
$$\times K[s,t_\Delta,(s_{\pi\Delta}\cdots)]. \quad (12b)$$

The function K in Eq. (12b) will be a known quantity. If one then substitutes empirical data, *at a fixed* t_Δ, for the left-hand side of Eq. (12b), the $(s_{\pi\Delta}\cdots)$ dependence of the ratio $K^{-1}d^2\sigma/dt_\Delta d(s_{\pi\Delta}\cdots)$ will provide a value for $\alpha_\pi(t_\Delta)$, at that t_Δ, which is independent of $s_{0\pi}$ and,

[18] W. Rarita *et al.*, Phys. Rev. **165**, 1615 (1968).

[19] At high-energy, the slope on a logarithmic plot of the πp elastic $d\sigma/dt_p$ is approximately the value 8.0 (GeV/c)$^{-2}$ used here; see K. J. Foley *et al.*, Phys. Rev. Letters **11**, 425 (1963); D. Harting *et al.*, Nuovo Cimento **38**, 60 (1965); Ref. 18.

[20] In employing $(s_{\pi\Delta}\cdots)^{\alpha_\pi}$ and $(s_{p\pi}\cdots)^{\alpha_P}$, the form given by the numerator of the $\cosh\xi_i$ variables of the Toller analysis (Ref. 1) is retained. See Eqs. (8) and (9). Especially, since the continuation is made here to near-threshold values of $s_{\pi\Delta}$, it is not clear that this choice is preferable to simply using $(s_{\pi\Delta})^{\alpha_\pi}$.

[21] Actually, most researchers studying quasi-two-body interactions concern themselves only with small values of t and extract factors such as $(\alpha+1)(\alpha+2)$ from the $[\Gamma(1+\alpha)]^{-1}$. For examples, see Ref. 18, and F. Arbab and C. B. Chiu, Phys. Rev. **147**, 1047 (1966).

moreover, of the parametrization of $G_\pi(t_\Delta)$. As remarked, the limited statistics precluded using this procedure except as a consistency check.

In view of the fact that one may express $(s_{0\pi})^{-2\alpha_\pi}$ as $\exp[-2(t_\Delta-m_\pi^2)\alpha_\pi' \ln s_{0\pi}]$, it is evident that changing $s_{0\pi}$ will serve to modify an inherent exponential damping in the variable t_Δ. Varying α_π' will directly affect the shapes of distributions in both $s_{\pi\Delta}$ and t_Δ. It is to be expected, therefore, that for a limited statistics sample of data various pairs of the quantities α_π' and $s_{0\pi}$ will generate successful fits. As will be stressed later, however, the data do require that $\alpha_\pi' \approx 1.0$ $(\text{GeV}/c)^{-2}$; an elementary pion-exchange calculation fails in a well-defined manner, which supports the argument given here that the amplitude representing the data must contain a factor such as $(s_{\pi\Delta})^{\alpha_\pi}$ with $\alpha_\pi < 0$. These issues are discussed further in Sec. IV F.

C. Normalization

The normalization factor N_0 of Eq. (7) was determined by factoring the diagram of Fig. 3 (a) about the pion-exchange line and considering the limit $t_\Delta \to m_\pi^2$. The contribution to N_0 of the $p\Delta\pi$ vertex, in that limit, may be evaluated in terms of the width Γ and mass m_Δ of the Δ^{++} as the factor[22]

$$g_\Delta^2 = 32\pi\Gamma m_\Delta^3 \{[(m_\Delta+m_p)^2-m_\pi^2]$$
$$\times[(m_\Delta-m_p)^2-m_\pi^2]\}^{-1/2}. \quad (13)$$

The remainder of the diagram represents off-mass-shell πp elastic diffraction scattering as may be recognized if Eq. (7) is rewritten as

$$\sum |\mathfrak{M}|^2 = g_\Delta^2 G_\pi(t_\Delta)\left(\frac{(s_{\pi\Delta}\cdots)}{s_{0\pi}}\right)^{2\alpha_\pi}(\sum|M_{\pi p}'|^2), \quad (14)$$

where $\mathfrak{M}_{\pi p}'$ is the amplitude for off-shell πp elastic diffraction scattering. In the diffraction-peak approximation,[19]

$$\sum |\mathfrak{M}_{\pi p}'|^2 \approx (\sum|\mathfrak{M}_{\pi p}'|^2)|_{t_p=0}e^{8t_p}. \quad (15)$$

Moreover, by using the optical theorem and assuming a purely imaginary amplitude at $t_p=0$, as is the case for P exchange, one may derive the expression[23]

$$\sum|\mathfrak{M}_{\pi p}'|^2 \approx [s_{p\pi}-(m_p+m_\pi)^2][s_{p\pi}-(m_p-m_\pi)^2]$$
$$\times \sigma_{\text{tot},\pi p}^2 e^{8t_\Delta}, \quad (16)$$

where $\sigma_{\text{tot},\pi p}$ is the πp total cross section. After combining Eqs. (14) and (16), and comparing the result

with Eq. (7) (at large $s_{p\pi}$ and small t_p and t_Δ), one obtains[24]

$$N_0 = 2g_\Delta^2 \sigma_{\text{tot},\pi p}^2. \quad (17)$$

A factor of 2 was inserted into Eq. (17) because antisymmetrization of the reaction amplitude is required, since the incident particles are identical. This effect is properly obtained by adding to Fig. 3(a) a diagram in which the incident particle momenta are interchanged. Due to the peripheral nature of the process, however, interference between the two diagrams is entirely negligible.

D. Data

A few remarks concerning the experimental distributions are in order before actual fits are described. When the mass spectrum of the final $(p\pi^+)$ combination of data from the four-prong reaction $pp \to pp\pi^+\pi^-$ is plotted, a strong Δ^{++} signal is evident. However, if one selects just those events from the broad enhancement $M(p\pi^+\pi^-) < 1600$ MeV, and then displays the $(p\pi^+)$ mass spectrum, it is less clear that a prominent Δ^{++} component is present, partially because of limited statistics and partially because of the fact that the three-body mass selection already kinematically constrains the two-particle mass to overlap the Δ^{++} band. This ambiguity is disturbing because there are at least two theoretical exchange mechanisms, alternative to $p\pi^-\Delta^{++}$, which may also serve to generate a broad low-mass $(p\pi^+\pi^-)$ enhancement. One involves the assumption that the process is actually $pp \to pp\sigma$, where σ is an s-wave two-pion resonance.[25] Another is based on the fact that the yield of a triple-exchange process such as the one diagrammed in Fig. 3(b) will be largely in the region of small $(p\pi^+\pi^-)$ mass values. In an attempt to overcome both of these objections, the assertion made here is that if a relatively narrow Δ^{++} mass definition is used $[M(p\pi^+)=1238\pm60]$, one will be left with data in which the true Δ^{++} to background signal is high. Note that the cut discussed earlier, $M(p\pi^-) > 2.0$ GeV, will remove double isobar events of the $N^{*0}\Delta^{++}$ variety.

The data on the final state $p\pi^-\Delta^{++}$ thus derived from $pp \to pp\pi^+\pi^-$ is produced highly peripherally with small momentum transfers to the Δ^{++} and to the p. The experimenters have noted, however, that there is a biased loss from the present sample of those events having $|t_p|$ very small and associated with the $(\Delta^{++}\pi^-)$ being in the forward hemisphere.[13] This loss is serious primarily for $|t_p| < 0.04$ $(\text{GeV}/c)^2$, and consequently, rather than limiting the study to the smaller, unbiased sample with $(\Delta^{++}\pi^-)$ backwards, all events were retained.

[22] This may be derived by writing the invariant amplitude for the $p\pi\Delta$ coupling, in Fig. 3(a), as $M = G\bar{u}_\nu(q_2)[p_2-q_2]_\nu u(p_2)$, where $u_\nu(q_2)$ is a 16-component Lorentz-vector Dirac spinor, and then evaluating G in terms of the width of the Δ. See, e.g., J. D. Jackson and H. Pilkuhn, Nuovo Cimento **33**, 906 (1964).

[23] Consult, e.g., G. Källén, *Elementary Particle Physics* (Addison-Wesley Publishing Co., Inc., Reading, Massachusetts, 1964), p. 18.

[24] If σ_{tot} in Eq. (16) is expressed in mb, and all other units are in GeV, in order for Eqs. (6) and (7) to yield a cross section in mb, an additional conversion factor of 2.5 mb^{-1} GeV^{-2} must be supplied on the right-hand side of Eq. (16).

[25] The mass of the σ would have to be 450 MeV or less, however.

E. Fit to Data

Attention was focused principally on the limited high-statistics region defined by $s_{p\pi} \geqslant 4.0$, $s_{\pi\Delta} \leqslant 5.0$, $|t_\Delta| \leqslant 0.5$, $|t_p| \leqslant 0.5$ (all units GeV²), where effects from diagrams other than the dominant one are expected to be negligible. Various pairs of the parameters α_π' and $s_{0\pi}$ in Eq. (7) provide acceptable fits in the sense of generating sets of singly differential distributions whose shapes agree with those of corresponding experimental distributions. A maximum-likelihood fit to the data, in the four-dimensional space of the two subenergy and two momentum-transfer variables, produced the values $\alpha_\pi' = 0.8 \pm 0.2$ GeV^{-2} and $s_{0\pi} = 0.50 \pm 0.05$ GeV²; however, the limited number of events involved and the biases discussed earlier reduce confidence in this determination.[26] As judged by eye, a somewhat better fit to the shapes of the several distributions results from the choice of $\alpha_\pi' = 1.2$ GeV^{-2} and $s_{0\pi} = 0.7$ GeV². Figures
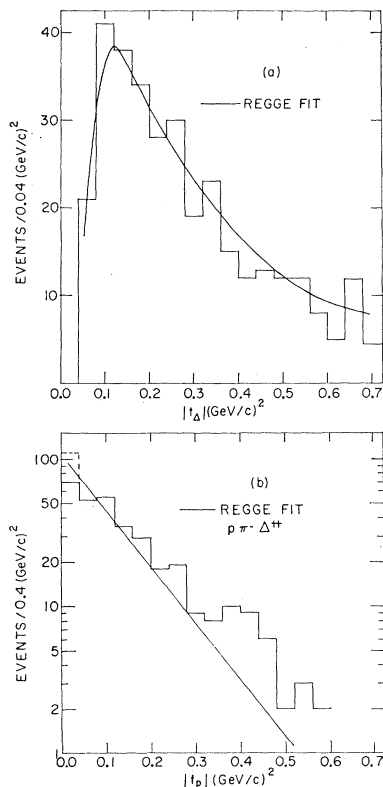

FIG. 4. (a) Distribution in the invariant four-momentum transfer squared to the Δ^{++} for the reaction $pp \to p\pi^-\Delta^{++}$ at 28.5 GeV/c. In the plot, there are 445 events for which $s_{p\pi} > 4.0$ GeV². The solid curve is the Regge-model fit with $\alpha_\pi' = 1.2$ GeV^{-2} and $s_{0\pi} = 0.7$ GeV². (b) Distribution in the invariant four-momentum transfer squared to the final p for the process $pp \to p\pi^-\Delta^{++}$, for events in the region $s_{p\pi} > 4.0$ GeV² and $|t_\Delta| < 0.8$ (GeV/c)². In the first bin, the dashed line corresponds to twice the number of those events associated with backwards $\Delta^{++}\pi^-$ production only.

---

[26] I am indebted to W. Edwin Ellis for assistance with the maximum-likelihood fit.

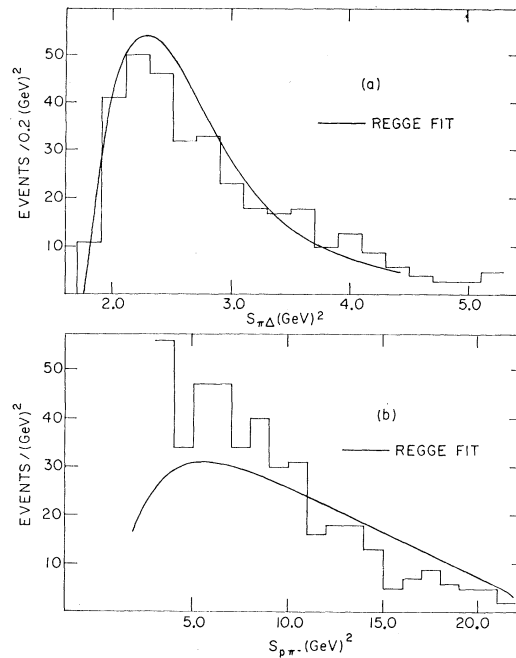FIG. 5. (a) Distribution in the invariant mass squared of the $(\Delta^{++}\pi^-)$ system containing events for which $s_{p\pi} > 4.0$ GeV² and $|t_\Delta| < 1.0$ (GeV/c)². (b) Distribution in the invariant mass squared of the $(p\pi^-)$ for events with $|t_\Delta| < 0.8$ (GeV/c)². The histogram is not plotted below $s_{p\pi} = 3.0$ GeV², which is outside the range of validity of the present calculation.

4–7 present computations of singly differential distributions using the latter pair of values. Naturally, the same selections were made in the computation as taken in the data; these are described in the captions. Good fits also resulted for the doubly differential distribution $d^2\sigma/dt_\Delta ds_{\pi\Delta}$, as well as for that in the variable $t_\pi = (q - p_2)^2$, which should be sensitive to the neglected baryon-exchange diagram's contribution. The values determined for the pair of quantities α_π' and $s_{0\pi}$ are well in line with the typical slopes and scale parameters of Regge theory and, moreover, are in close agreement with values obtained for the same parameters in a fit to data at the much lower momentum of 6.6 GeV/c.[6]

Because the constant term N_0 in Eq. (7) cancels out in the definition of the likelihood function, the maximum-likelihood fit is to the shape of the distribution of events in the $(s_{\pi\Delta}, s_{p\pi}, t_\Delta, t_p)$ space only and not to the absolute normalization. Similarly, the fits carried out by eye were to shapes only. Thus the degree of agreement between model and data in terms of absolute normalization is a measure of the extent to which the diagram of Fig. 3(a), indeed, dominates the reaction. The actual Regge-model curves shown in Figs. 4–7 were obtained after multiplying Eq. (7) by the factor 0.95, whereas the other pair of values ($\alpha_\pi' = 0.8$ GeV^{-2} and $s_{0\pi} = 0.5$ GeV²) yield a cross section which is 96% of the

experimental value.[27] Consequently, the absolute-normalization aspect of the model is remarkably good, as it was also at 6.6 GeV/c.[6]

F. Discussion of the Fit

It is evidently not possible to prove in a rigorous sense that these data (or any data in the physical region $t_\Delta < 0$) demand the existence of the pole singularity postulated here in $G_\pi(t_\Delta)$, Eq. (11), at the unphysical point $t_\Delta = m_\pi^2$. However, the good agreement in absolute normalization and the reasonableness of the values determined for α_π' and $s_{0\pi}$ demonstrate strong consistency of the pion-exchange aspect of the model with the experimental situation. On the other hand, it is important to inquire into the role played by Reggeization in this problem.

The finite, positive value for the slope α_π' of the trajectory of the exchange pion, operating through the factor $(s_{\pi\Delta})^{2\alpha_\pi}$ in Eq. (7), is indeed *essential* in order to generate a distribution in the $(\pi^-\Delta^{++})$ invariant mass
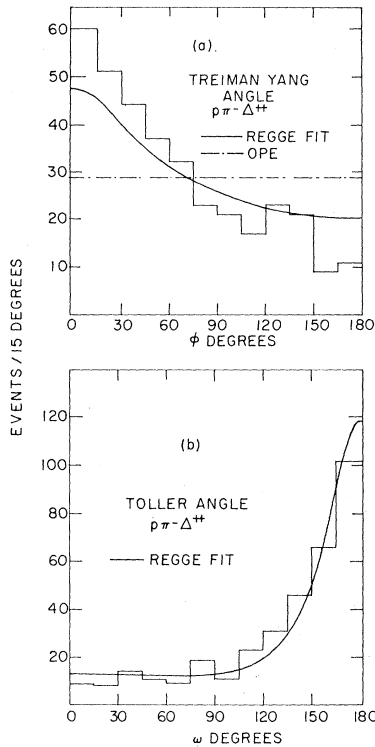

FIG. 6. (a) Distribution in the Treiman-Yang angle for $p\pi^-\Delta^{++}$ defined in the $p\pi^-$ rest frame as $\phi = \cos^{-1}[(\mathbf{p}_1 \times \mathbf{q}_1) \cdot (\mathbf{p}_2 \times \mathbf{q}_2)/ |\mathbf{p}_1 \times \mathbf{q}_1| |\mathbf{p}_2 \times \mathbf{q}_2|]$, where the $\mathbf{p}_i$ and $\mathbf{q}_i$ in this expression are three-vectors specified by reference to Fig. 3 (a). (b) Distribution in the Toller angle, $\omega = \cos^{-1}[(\mathbf{p}_1 \times \mathbf{q}_1) \cdot (\mathbf{p}_2 \times \mathbf{q}_2)/ |\mathbf{p}_1 \times \mathbf{q}_1| |\mathbf{p}_2 \times \mathbf{q}_2|]$, measured in the rest frame of the π^-. The Regge model fits the distribution without requiring explicit ω dependence in the matrix element. Both (a) and (b) contain 349 events for which $s_{p\pi} > 4.0$ GeV^2 and $|t_\Delta| < 0.8$ (GeV/c)^2.
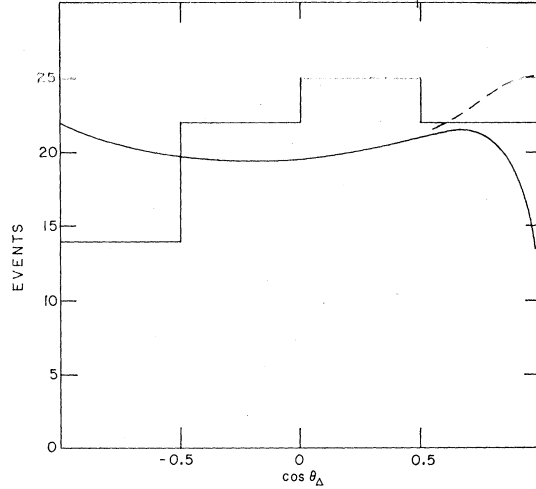

FIG. 7. Distribution in the cosine of the Δ^{++} production angle. With reference to Fig. 3(a), the angle is defined by $\cos\theta_\Delta = \mathbf{q}_2 \cdot \mathbf{p}_2/ |\mathbf{q}_2| |\mathbf{p}_2|$, where the three-vectors $\mathbf{q}_2$ and $\mathbf{p}_2$ are measured in the frame of reference in which the $(\pi^-\Delta^{++})$ system is at rest. The selections imposed here were $|t_p| \leqslant 0.3$ (GeV/c)^2, $M(\pi\Delta) \leqslant 1.6$ GeV, and $s_{p\pi} \geqslant 4.0$ GeV^2. The solid line is the Regge-model prediction; the dashed curve is the model prediction with the selection on $s_{p\pi}$ removed.

whose peak position and full width at half maximum are in agreement with the data, and in order to produce the type of asymmetry observed in the distribution in the Trieman-Yang angle, Fig. 6(a). These two features of the Reggeized approach may be recognized most directly by contrasting the form and results of this model with those of an elementary one-pion-exchange–diffraction-scattering (OPE-DS) or Deck model,[12] which the Reggeized model essentially becomes in the limit $\alpha_\pi' \rightarrow 0$.

In the OPE-DS model, one replaces Eq. (7) by[12,28]

$$\sum |\mathfrak{M}|_{\mathrm{OPE}}^2 = N_0(t_\Delta - m_\pi^2)^{-2}[s_{p\pi} - (m_p + m_\pi)^2]$$
$$\times [s_{p\pi} - (m_p - m_\pi)^2] \exp[8t_p + \lambda(t_\Delta - m_\pi^2)]. \quad (18)$$

In Eq. (18), N_0 has the same value given previously in Eq. (17), and the option of a form factor $\exp[\lambda(t_\Delta - m_\pi^2)]$ has been explicitly introduced (without it the experimental distribution in t_Δ cannot be fitted using OPE-DS). *At small values of* $|t_\Delta|$, *the essential difference between Eq. (18) and Eq. (7) is the presence in Eq. (7) of the factor* $(s_{\pi\Delta}\cdots)^{2\alpha_\pi}$.

At the energy being considered here, Eq. (18) will yield the same curve for the distribution in t_Δ as that given by the Regge model, shown in Fig. 4(a), if $\lambda \equiv 1.0$ GeV^{-2} in Eq. (18). With λ thus fixed at 1.0 GeV^{-2}, Eq. (18) was used to compute the expected OPE-DS distribution in the $\pi^-\Delta^{++}$ invariant mass. The results of the two models are compared in Table I. Whereas the Regge model is able to adequately fit the distributions in both t_Δ and $s_{\pi\Delta}$, the OPE-DS model

---

[27] In obtaining the normalization, the values $\Gamma = 120$ MeV and $\sigma_{\mathrm{tot},\pi p} = 28$ mb were used in Eqs. (12) and (16). See also Ref. 24. The data correspond to a cross section of 0.87 μb/event.

[28] Additional dependence on t_Δ, associated with the spin structure of the Δ^{++}, was evaluated at $t_\Delta = m_\pi^2$ and absorbed into N_0.

TABLE I. Comparison of Regge- and OPE-model predictions with data for the location (M_0) of the peak and full-width at half-maximum Γ of the distribution in the invariant mass of the $(\pi^-\Delta^{++})$ system, for two values of the cutoff in the integration over the variable t_Δ. The last line contrasts the predictions for the distribution in the Treiman-Yang angle. The OPE model contains a form factor, as described in the text, with $\lambda = 1.0$ $(\text{GeV}/c)^{-2}$.

	OPE-DS $(\lambda=1)$ (MeV)	Double Regge pole $(\alpha_\pi' = 1.2; s_{0\pi}=0.7)$ (MeV)	Experimental (MeV)
M_0; $\lvert t_\Delta \rvert < 0.5$	1500	1480	1480
M_0; $\lvert t_\Delta \rvert < 1.0$	1580	1520	1500
Γ; $\lvert t_\Delta \rvert < 0.5$	420	280	200
Γ; $\lvert t_\Delta \rvert < 1.0$	540	380	320
Treiman-Yang; $\lvert t_\Delta \rvert < 0.8$	Flat	41% asym.	See Fig. 6(a)

(even with form factor) can reproduce only one of the two.

Phase-space threshold effects, of course, compel the distribution in $s_{\pi\Delta}$ to vanish at threshold and, as discussed elsewhere, the peaking of the cross section at near threshold values of $s_{\pi\Delta}$ is largely a kinematic reflection of the suppression of large values of t_Δ and t_p in the doubly peripheral diagram. However, Reggeization enhances the low-mass peaking by directly providing an $s_{\pi\Delta}$-dependent factor in the matrix element whose effect is to further suppress large values of $s_{\pi\Delta}$. Analytically, this narrowing of the low-mass enhancement can be understood by observing that the $(s_{\pi\Delta})^{2\alpha_\pi}$ term in Eq. (7) is roughly approximate to $(s_{\pi\Delta})^{-0.3}$ because $\alpha_\pi = (t_\Delta - m_\pi^2)\alpha_\pi'$ and the peak in the t_Δ distribution occurs at $t_\Delta \approx -0.12$ $(\text{GeV}/c)^2$ [see Fig. 4(a)].

In the last row of Table I and in Fig. 6(a), the predictions of the two models for the distribution in the Treiman-Yang angle φ are contrasted. The OPE-DS model produces a perfectly flat distribution in the variable φ, whereas the Regge model yields an asymmetry about 90° and a peaking toward 0° similar to that observed in the data. In this work, the computed asymmetry about 90° is defined by

$$A = \frac{d\sigma/d\varphi(0°) - d\sigma/d\varphi(180°)}{d\sigma/d\varphi(0°) + d\sigma/d\varphi(180°)}. \quad (19)$$

The physical reason for the large asymmetry developed by the Regge model rests simply, again, in the factor $(s_{\pi\Delta})^{2\alpha_\pi(t_\Delta)}$; this coupling of the adjacent momentum-transfer t_Δ and subenergy $s_{\pi\Delta}$ variables may be reexpressed in terms of momentum variables and implies a correlation of the momenta of the particles in the final state. Stated otherwise, the sets $(s, s_{p\pi}, t_\Delta, t_p, s_{\pi\Delta})$ and $(s, s_{p\pi}, t_\Delta, t_p, \varphi)$ are two complementary sets of independent variables in terms of which the amplitude may be expressed. They are related through an expression of the form

$$s_{\pi\Delta} = A(t_\Delta, t_p, s_{p\pi}) + \cos\varphi \, B(t_\Delta, t_p, s_{p\pi}), \quad (20)$$

which shows that a dependence in the amplitude on

$s_{\pi\Delta}$ leads, in general, to a nonisotopic φ distribution. Absence of $s_{\pi\Delta}$ or φ dependence in $(\sum \lvert \mathfrak{M} \rvert^2)$ automatically leads to a perfectly flat distribution in the φ variable.

The asymmetry about 90° in the φ distribution, quite sensitive to the value of α_π', increases as α_π' increases. Moreover, for fixed α_π', it is also an increasing function of the variable $\lvert t_\Delta \rvert$ because $\lvert \alpha_\pi \rvert$ increases with $\lvert t_\Delta \rvert$; for example, if instead of restricting the t_Δ integration to values $\lvert t_\Delta \rvert \leqslant 0.8$ $(\text{GeV}/c)^2$, as in Fig. 6(a) and Table I, one chooses the cut $\lvert t_\Delta \rvert \leqslant 0.3$ $(\text{GeV}/c)^2$, the predicted asymmetry is reduced to 28%. On the other hand, a fixed constant-power behavior in the variable $s_{\pi\Delta}$ would lead to an asymmetry which is independent of t_Δ. The data at both 6.6 $\text{GeV}/c^{6,29}$ and 28.5 GeV/c give definite evidence for an increase in the asymmetry as one takes progressively larger cuts in $\lvert t_\Delta \rvert$; the data are thus highly suggestive of a mechanism whereby the spin of the exchanged object is a function of t_Δ.

The validity of the conclusions presented in the above paragraphs does not rely upon the very simple form chosen for the t_Δ dependence of $(\sum \lvert \mathfrak{M} \rvert^2)$, namely, the function $G_\pi(t_\Delta)$ in Eq. (11). Other, more complicated, expressions (for example, $G_\pi(t_\Delta)$ multiplied by $[\Gamma(1+\alpha)]^{-2}$) were tried, with the result that although $s_{0\pi}$ is forced to change, acceptable fits to the $s_{\pi\Delta}$ and φ distributions definitely require that $\alpha_\pi' \approx 1.0$ $(\text{GeV}/c)^{-2}$. The criticism may be leveled, of course, that one is misapplying the pion-exchange concept when using it as far out in t_Δ as $\lvert t_\Delta \rvert \approx 0.5$ $(\text{GeV}/c)^2$. However, even when one restricts the analysis to $\lvert t_\Delta \rvert < 0.2$ $(\text{GeV}/c)^2$ $= 10 m_\pi^2$, the experimental anisotropy in the φ distribution is evident and requires that $\alpha_\pi' \approx 1.0$ for its interpretation within the context of this model. It would be valuable to pursue this matter of the pion-trajectory parameters by analyzing high-statistics data on the several reactions listed at the beginning of Sec. IV.

In the remainder of this section, additional comments are made regarding the distributions presented in Figs. 4–7. The discrepancy between the Regge fit and the data for values of $\lvert t_p \rvert < 0.04$ $(\text{GeV}/c)^2$, shown in Fig. 4(b), is attributable to the bias, discussed earlier, which results in a loss of events at small $\lvert t_p \rvert$ when the $(\Delta^{++}\pi^-)$ system is in the forward hemisphere. Indeed, the dashed histogram in the region $\lvert t_p \rvert < 0.04$ $(\text{GeV}/c)^2$ corresponds to twice the number of events present in the backwards $(\Delta^{++}\pi^-)$ sample; this bias is not serious in the remainder of the $\lvert t_p \rvert$ distribution. Because events with $\lvert t_p \rvert < 0.04$ $(\text{GeV}/c)^2$ occur preferentially with small values of $s_{\pi\Delta}$, this bias explains, in part, the apparent normalization discrepancy in the distribution in the $s_{\pi\Delta}$ variable, shown in Fig. 5(a). The effects of the bias on the other distributions is negligible. The distribution in the $s_{p\pi}$ variable [Fig. 6(b)] shows that the $(s_{p\pi}\cdots)^2$ factor in Eq. (7), characteristic of Pomeranchon exchange or diffraction scattering, is a

29 The definition of φ used in Ref. 6 differs slightly from that in this work; they are related by $\varphi(6.6) = \varphi(28.5) + \pi$.

good representation of the over-all $s_{p\pi}$ dependence for values $s_{p\pi} \geqslant 4.0$ GeV^2, as asserted earlier. The agreement at the lower end of the scale can be improved by the addition of a diagram with P' exchange, for example; moreover, a mild slope[18] ($\alpha_p' \approx 0.3$) for the Pomeranchuk trajectory results in better agreement of the shape of the calculated distribution with the data at the upper end of the spectrum. The agreement evident in Fig. 6(b) suggests that the choice made here to ignore explicit ω dependence in the two-Reggeon–one-particle vertex is quite consistent with the experimental situation.

The severe cuts $|t_p| < 0.3$ (GeV/c)^2 and $M(\pi\Delta) < 1.6$ GeV in Fig. 7 were chosen in order to focus on the region of the low-mass ($\pi\Delta$) enhancement. The additional cut $s_{p\pi} \geqslant 4.0$ GeV^2, necessary in order to eliminate double-isobar events, removes events in the region $\cos\theta_\Delta > 0.5$. The data show evidence for a depletion of events for $\cos\theta_\Delta < -0.5$; these events are statistically associated with values $|t_\Delta| > 0.5$ (GeV/c)^2. The discrepancy between the calculated curve and the data for $\cos\theta_\Delta < -0.5$ is perhaps indicative of interference between the principal diagram, Fig. 3(a), and others (such as the Δ^{++} exchange possibility discussed earlier) in the region $|t_\Delta| > 0.5$ (GeV/c)^2. Both the calculated curve and the data become peaked towards $\cos\theta_\Delta = +1$ for larger values of $M(\pi\Delta)$.

G. Conclusions

The over-all fit to the data supports the double-Regge-pole-exchange approach to three-particle production and, in particular, calls for the exchange of a Reggeized pion whose average slope in the momentum-transfer range $0 > t > -0.5$ (GeV/c)^2 is approximately 1.0 GeV^{-2}. Diagrams other than the dominant one shown in Fig. 3(a) provide negligible contributions to the process $pp \to p\pi^-\Delta^{++}$, at least in the limited domain defined at the beginning of Sec. IV E. The low-mass ($\pi^-\Delta^{++}$) enhancement and related distributions are adequately fitted without requiring any explicit ($p\pi^+\pi^-$) resonance production. The neglect of spin effects in the spin-average approach taken here, especially as regards the Δ^{++}, would not appear to alter these conclusions in view of the fact that the coupling of the pion to the $\frac{3}{2}$ helicity states of the Δ is known from quasi-two-body final-state data to be a small effect.[30]

V. FEATURES OF THE MODEL

The discussion in Sec. IV F regarding the $s_{\pi\Delta}$ and φ distributions underscored some general aspects of the Regge-model matrix element and fits. In this section, a few additional features will be stressed.

A. Slopes in the Momentum-Transfer Variable

A particularly interesting consequence of the doubly peripheral approach becomes evident when one ex-

amines the distribution in the variable t_p for various choices of the invariant mass $M_{\pi\Delta}$ of the ($\pi^-\Delta^{++}$) combination. In a missing-mass type of counter experiment, these are essentially the two relevant variables when one triggers on a fast proton[31]; they are also, of course, the usual variables in a Chew-Low plot. The calculated doubly differential distribution $d^2\sigma/dt_p dM_{\pi\Delta}$ is in fact well approximated in the region $|t_p|$ 0.3 (GeV/c)^2 by the expression

$$d^2\sigma/dt_p dM_{\pi\Delta} \approx \exp(a+bt_p), \qquad (21)$$

where the parameters a and b, although independent of t_p, depend strongly on $M_{\pi\Delta}$. In Table II, values of b for various $M_{\pi\Delta}$ are presented. Also given in this table are the calculated values of b versus $M(n\pi^+)$ for the recently published study of $pp \to pn\pi^+$ at 28.5 GeV/c.[7]

It will be noted that for values of $M_{\pi\Delta}$ or $M_{\pi n}$ near their respective thresholds the calculated (output) value of b is twice the elastic-scattering value of 8.0 (GeV/c)$^{-2}$ used as input in Eq. (7). This substantial increase may be described as kinematical "feed-through" from the right to the left in the diagram of Fig. 3(a); any dependence on the variable t_Δ in Fig. 3(a) reflects itself kinematically as a dependence on the variable t_p. It is the steeply falling $(t_\Delta - m_\pi^2)^{-2}$ character of the t_Δ variation in either Eq. (7) or Eq. (18) which tends to force the t_p dependence to be sharper than the input e^{8t_p} variation. The dependence in the matrix element on the subenergy variables, however, subsequently modulates this effect and contributes to the variation of b with $M_{\pi\Delta}$.

Such rapid variation of the slopes (on logarithmic plots) of momentum-transfer distributions with the mass of produced states is a general feature of the double-peripheral approach to three-particle production. Effects of this type have been observed experimentally, notably in the missing-mass type of counter experiments[31] and also by several bubble-chamber groups.[32,33] Walker, in particular, has described such a variation as an important aspect of the diffraction-dissociation mechanism.[33,34] It is evident that one must be cautious in interpreting such systematic variation of the momentum-transfer dependence with invariant mass; the above analysis shows that the effect is not obviously correlated with resonant behavior. It is the pion-pole term in the t_2 variable that causes the extra steep fall-off in t_1.

B. Energy Dependence

Subject to the provision that the $s_{p\pi}$ integration is carried out over its full range, the Regge-model matrix

[30] G. Fox (private communication).

[31] E. W. Anderson et al., Phys. Rev. Letters **16**, 855 (1966); G. Belletini et al., Phys. Letters **18**, 167 (1965); J. M. Blair et al., Phys. Rev. Letters **17**, 789 (1966); K. J. Foley et al., ibid. **19**, 397 (1967).
[32] J. Bartsch et al., Phys. Letters **27B**, 336 (1968).
[33] W. D. Walker et al., Phys. Rev. Letters **20**, 133 (1968); A. F. Garfinkel et al., University of Wisconsin Report, 1968 (unpublished).
[34] M. L. Good and W. D. Walker, Phys. Rev. **120**, 1857 (1960).

$pp \to p\pi^-\Delta^{++}$		$pp \to p\pi^+n$	
$M(\pi\Delta)$ (GeV)	b (GeV/c)$^{-2}$	$M(\pi n)$ (GeV)	b (GeV/c)$^{-2}$
1.40	14.0	1.10	18.0
1.50	11.5	1.20	12.5
1.60	10.0	1.30	9.5
1.80	8.0	1.50	7.0
2.00	5.2	1.80	4.5

element, Eq. (7), yields a distribution in the $(\pi^-\Delta^{++})$ invariant mass whose peak position and full-width at half-maximum are essentially independent of incident energy from 6.0 to 100 GeV/c. The total cross section is likewise approximately energy-independent. Similar statments apply to the calculation of $pp \to pn\pi^+$ reported earlier.[7]

C. Cuts in $s_{p\pi}$

The restriction in Sec. IV to the study of only those events for which $s_{p\pi} \geqslant 4.0$ GeV^2 is by no means fundamental. However, if one desires to represent data at smaller values of the $s_{p\pi}$ variable, then diagrams in addition to that given in Fig. 3(a) must be included in the computations; these would include (but are not limited to) diagrams of the same topology as that given in Fig. 3(a) but with P replaced by P' or ρ, for example. Evidently this argument does not imply the absence of a P contribution below $s_{p\pi} = 4.0$ GeV^2; rather, it suggests, supported by Fig. 4(b), that other effects are more prominent there. By the same token, if one chooses to represent the data by using only the diagram shown in Fig. 3(a), whether in the Regge form or in the non-Reggeized OPE-DS[12] model, then either a cut in the $s_{p\pi}$ variable is in order or some other subtraction should be made to eliminate the non-Pomeranchon effects. This last point is often ignored by researchers who use a non-Reggeized version of Fig. 3(a) for the purpose of calculating "background" contributions to processes such as $\pi p \to A_1 p$ or $Kp \to K^{**} p$.

D. Direct-Channel Diagrams

By way of emphasizing that the model discussed in this paper is one which seeks to interpret peripheral three-particle final-state processes entirely in terms of *cross-channel exchange* contributions, it should be noted that it is inappropriate to consider including a diagram of the direct-channel variety, representing pole-term propagators in one of the *subenergy* variables [for examples, see Fig. 1(A) of the Ross-Yam analysis[12] or Fig. 6(b) of the Resnick paper[12]]. For the $pp \to p\pi^-\Delta^{++}$ process studied in this paper, a direct-channel diagram, of the type to which reference is being made here, is one representing pp diffraction scattering with one of the final-state (off-mass-shell) protons subsequently dissociating into a $(\Delta^{++}\pi^-)$ pair.

To include such a diagram along with that of Fig. 3(a) would likely result in the same type of double counting that is involved in two-body processes if one adds direct-channel resonance terms to a so-called background formed of Regge-pole-exchange contributions. This last remark is qualitatively evident if one regards the $\pi^-\Delta^{++}$ final state in Fig. 3(a) as resulting from the "quasi-two-body" interaction between the Pomeranchon and the proton p_2.

VI. DISCUSSION

In several recent analyses, a double-Regge-pole-exchange approach of the type described here has provided good agreement with all experimental distributions even at near-threshold values of the invariant mass of a final pair of particles in a three-body final-state process. In addition to the proton-proton interaction studies already mentioned, in which data in the region of the 1400-MeV nucleon-isobar-type enhancement has been fit, data from the reactions $\pi p \to \pi\rho p$ at 13.0 and 20.0 GeV/c[9] and coherent $\pi d \to \pi\rho d$ at 8.0 GeV/c[8] are also in agreement with calculations based on a Reggeized pion–Pomeranchon doubly peripheral diagram. These last two reactions give evidence for a broad A_1 (mass ≈ 1100 MeV) enhancement in the $\pi\rho$ mass spectrum which the model adequately describes. Moreover, a partial-wave analysis of the Regge-model amplitude predicts that 85% of the calculated A_1 enhancement is an s-wave $(l=0)$ $J^P = 1^+$ $(\pi\rho)$ system, in good agreement with the fraction determined experimentally in the 8.0-GeV/c study.[8]

It is inappropriate, however, to conclude from the results of these studies that either the A_1 or the $N^*(1400)$ effects observed in the above reactions are not resonances in the usual sense. For, whereas there has been no conclusive demonstration as yet that a cross-channel Regge-pole-exchange description and a direct-channel resonance interpretation are perfectly dual[11] alternative modes of understanding, a growing body of information points in that direction. In quasi-two-body scattering, the finite-energy sum-rule[10] approach has demonstrated that the appropriate sum over Regge-pole-exchange contributions in the cross channel provides an adequate semilocal-average description of the *imaginary* part of the scattering amplitude, expressed in terms of direct-channel resonance contributions. Moreover, partial-wave analyses performed on the (asymptotically valid) Regge form of the amplitude and accepted even in the intermediate-energy resonance domain have demonstrated that resonance like Argand-diagram circles are present in the Regge amplitude.[35]

Based on the fact that no poles in the direct-channel energy variable are present in the asymptotic Regge

[35] C. Schmid, Phys. Rev. Letters 20, 689 (1968); P. D. B. Collins, R. C. Johnson, and E. Squires, Phys. Letters 27B, 23 (1968); C. B. Chiu and A. Kotanski, CERN Report Nos. 907 and 939, 1968 (unpublished); V. Alessandrini et al., CERN Report No. 922, 1968 (unpublished).

amplitude, objections have been raised to the interpretation of these circles as manifestations of resonances. Schmid countered these objections using an analogical reference to the Stirling approximation to the Γ function[35]; however, a precise judgment on the duality issue awaits the development of a unique representation of the scattering amplitude, valid at all energies and having Regge asymptotic behavior. Some progress in this regard is evident in a recent proposal by Veneziano.[36]

The relevance of this discussion to three-particle final-state problems is evident upon observing that insofar as the final state $\pi^-\Delta^{++}$, for example, is concerned, it can be regarded as being the end product in Fig. 3(a) of a quasi-two-body proton-Pomeranchon interaction, with pion exhcange being the dominant cross-channel exchange mechanism. Thus, in effect, Reggeization has introduced a third, but possibly unifying, interpretation to the yet unsettled resonance versus kinematic enhancement question surrounding the experimental status of the A_1, $N^*(1400)$, and $K\pi\pi\,Q$ bump[37] effects. This controversy has the practical consequence of making it difficult to quote unambiguously values for "resonance" cross sections in these mass regions.

[36] G. Veneziano, Nuovo Cimento **57A**, 189 (1968).
[37] G. Goldhaber *et al.*, Phys. Rev. Letters **19**, 972 (1967), and references therein.

The theoretical status of the calculations described here is somewhat primitive. Most fundamental, of course, is the matter of the use, at nonasymptotic values of $s_{\pi\Delta}$, of the unmodified Regge form of the amplitude given in Eq. (7). Present justification relies on duality-type arguments[11] and on the not inconsiderable agreement with data on different reactions over a wide energy range. Further progress in this regard awaits the development, for example, of a Veneziano[36] type of representation for five-body processes which would have Regge asymptotic behavior in the two subenergy variables but would provide a better near-threshold dependence on these variables.

ACKNOWLEDGMENTS